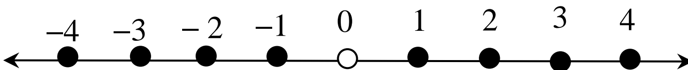
3. INEQUALITIES AND ABSOLUTE VALUES

§3.1. The Ordering of the Real Numbers

In addition to the arithmetic structure of the real numbers there's the order structure. The real numbers can be represented by points on a line and if one point is to the left of another we say that the corresponding number is smaller, or less, than the other.



Alternatively we can call the numbers on the right of zero the **positive** numbers and then define $\boldsymbol{x < y}$ if $y = x + z$ for some positive z. Numbers to the left of zero are called **negative**.

We define $\boldsymbol{x \le y}$ to mean that $x < y$ or $x = y$. Of course if x and y were specific numbers we'd know which is the case and would write $x < y$ or $x = y$ instead of the more uncertain $x \le y$. But often we don't know which is the case.

Sometimes it's convenient to specify the larger number first and write $\boldsymbol{x \ge y}$ instead of $y \le x$. Finally, $\boldsymbol{x > y}$ means the same as $y < x$.

Example 1: $2 \le 4$. In fact we can be more specific by writing $2 < 4$. These can be written alternatively as $4 \ge 2$ and $4 > 2$ respectively.

Inequalities look rather like equations and in many ways we can operate with them the way we would equations. But beware. We'll later see that there are some important differences.

The relation $\le$ has three important properties that are obvious when you consider the number line.

Reflexive Property: $x \le x$ for all x.
Anti-Symmetric Property: If $x \le y$ and $y \le x$ then $x = y$.
Transitive Property: If $x \le y$ and $y \le z$ then $x \le z$.

The first of these is so obvious it's barely worth mentioning. The second is occasionally useful as a way of proving that two numbers are equal. The third is something we use all the time.

There are other orderings in mathematics where these three properties hold. For example, if S and T are sets the statement S is a **subset** of T means that every element of S is an element of T. We write $S \subseteq T$. For example the set of boys in a family is a subset of the set of children in the family.

The analogy is made clear by the fact that the symbol we use for subset is very similar to the one we use for ‘less-than-or-equals’. Moreover the reflexive, anti-symmetric and transitive properties hold for subsets.

Reflexive Property: $S \subseteq S$ for all S.
Anti-Symmetric Property:
If $S \subseteq T$ and $T \subseteq S$ then $S = T$.
Transitive Property: If $S \subseteq T$ and $T \subseteq U$ then $S \subseteq U$.

Another important property of the ordering of real numbers is the following.

Total Order Property: For any two distinct real numbers either $x < y$ or $y < x$.

The real numbers are arranged linearly and any two such numbers can be compared. Notice that the subset ordering doesn’t have this property. It is possible to have two distinct sets S, T where neither is a subset of the other. For example, the set of positive numbers is not a subset of the set of negative numbers, nor vice versa.

§3.2. Sets

We often need to specify collections of numbers, or indeed many other types of mathematical objects. We define a **set** to be any collection of things. Of course this

is not a very satisfactory definition because ‘collection’ is a synonym for ‘set’ and we may well ask, “what is a collection?”

We can give many examples of sets and hope that we have some sort of inbuilt notion of ‘set’, hardwired into our brains. It is certainly not possible to define a set in terms of more primitive concepts.

If ever you get to investigate the foundations of mathematics in a wholly rigorous way you will find that ‘set’ is taken to be an undefined concept, with various axioms regulating the use of the word. But at your stage you just have to rely on your innate concept that you somehow acquired, or perhaps it was there when you were born, before you had a word for it.

One important thing that has to be pointed out. In normal usage, ‘set’ suggests that the objects that are in a set must all match each other in some way, as in a set of cups and saucers. But the mathematical usage of the word ‘set’ allows for the objects to be wildly different.

In his *Alice in Wonderland*, Lewis Carroll, who was a mathematics lecturer at Oxford, emphasises this in the *Song of the Walrus*.

> The time has come,' the Walrus said,
> To talk of many things:
> Of shoes — and ships — and sealing-wax —
> Of cabbages — and kings.

What a disparate collection of things, yet one can consider the set that contains all shoes, all ships, all pieces of sealing-wax, a whole lot of cabbages, and even some kings.

The objects that are contained in a set are called **elements**. And if element x belongs to set S, we write $x \in S$ to indicate that x is an element of S. If it isn't we write $x \notin S$.

Now it's common to use lower case letters for elements and capital letters for sets, but remember that one can have sets of sets, and sets of sets of sets, so the distinction between elements and sets can't be maintained. But, at our level, we won't encounter this problem.

Example 2: If S is the set of all even numbers then $6 \in S$, while $7 \notin S$. And please note that $0 \in S$. I have had many arguments with teachers who claim that 0 is neither odd nor even, confusing the situation with positive and negative numbers, where 0 is neither odd nor even.

If a set is finite we can list the elements and we write **{a, b, c}** for the set that contains just a, b and c. We also write $\{x_1, x_2, \ldots, x_n\}$ for the set that consists of precisely these n elements.

We can even extend this notation to certain infinite sets, where we list the first few elements and where there's an obvious pattern whereby we can extend the list.

Example 3: {3, 18, 25} is the set with the three elements 3, 18 and 25.
{1, 2, 3, …} indicates the set of all positive integers and {0, 1, 4, 9, …} suggests that it is the set of all perfect squares. But {3, 18, 25, …} is ambiguous because there is no obvious way of continuing the sequence.

I might have given the impression that the elements of a set are ordered in some way, and that we can talk about the first element in the set. However a set is without order. The elements can be written in any order and it's still the same set. So {3, 1, 2} is the same set as {1, 2, 3}. If we wish there to be a specific ordering we have to use the word 'sequence'. A **sequence** is a set, together with a specific ordering. We write a sequence as $(x_1, x_2, \ldots, x_n)$ or $(x_1, x_2, \ldots)$ in the case of an infinte sequence.

But infinite sets are too big for us to list all the elements and some sets are even too big for this to be done in theory.

For example, take the set of all positive real numbers. Which is the smallest? Even if we decided we would list them according to something other than size, it can be shown that it is impossible to write the set of all positive real numbers in an infinite list.

To get around this problem we have an alternative method of describing a set. If P is some property, so that Px is shorthand for the statement ‘x has the property P’, then we write the set of all things that have the property P as **{x | Px}**. We read this as “the set of all *x* such that Px is true”. In practice we work within some **universal set**, such as the set of all real numbers. It is important to know what the universal set is at all times.

For example, if $S = \{x \mid x = y^2 \text{ for some } y\}$, the set of all squares, then:

$3 \notin S$ if the universal set is the set of integers,

$3 \in S$ if the universal set is the set of real numbers,

$-1 \notin S$ if the universal set is the set of real numbers,

$-1 \in S$ if the universal set is the set of complex numbers.

Example 4: $\{x \mid x > 0\}$ denotes the set of all positive real numbers, where our universal set is the set of all real numbers. If our universal set was the set of all complex numbers this wouldn’t make sense because we can’t define inequalities for complex numbers.

There are some very deep logical difficulties with the free use of this concept of turning properties into sets. It shouldn’t worry you, at your level of mathematics, but just for interest, you may be interested in learning about the Russell Paradox.

Let our universe consist of all sets and let

$$S = \{x \mid x \notin x\}.$$

It may seem strange for a set to be an element of itself, though if x was allowed to be the set of all sets we would have $x \in x$.

Now if $S \in S$ it would have to have the property that defines S, and so $S \notin S$, a contradiction.

If $S \notin S$ it doesn't have the property that defines S. Therefore $S \in S$, again a contradiction.

Contradictions are useful in mathematics because they channel us into the correct conclusion, as in Proof By Contradiction. But an out-and-out contradiction cannot be allowed in mathematics because if you did you could prove anything.

There is a way out of such difficulties. This is to be very careful as to what properties can be allowed to give rise to a set. For example the collection of all sets cannot be considered as a set. If you are interested in such things you can read about them in my notes on *Set Theory*. But in doing 'ordinary mathematics' we will never encounter such exotic difficulties.

If S, T are sets their **intersection** is the set of all things common to both, that is:

$$\mathbf{S} \cap \mathbf{T} = \{x \mid x \in S \text{ and } x \in T\}.$$

Moreover their **union** is the set of all things that belong to at least one of S, T, that is:

$$\mathbf{S} \cup \mathbf{T} = \{x \mid x \in S \text{ or } x \in T\}.$$

Here we use the mathematical meaning of ‘or’ that includes the possibility of both.

Their difference is the set of all things that belong to one set but not the other, that is:

$$\mathbf{S - T} = \{x \mid x \in S \text{ and } x \notin T\}.$$

Examples 5: If $S = \{1, 3, 7, 8\}$ and $T = \{3, 4, 5, 6, 7\}$ then $S \cap T = \{3, 7\}$, $S \cup T = \{1, 3, 4, 5, 6, 7, 8\}$.
$S - T = \{1, 8\}$ while $T - S = \{1, 4, 5, 6, 8\}$.
$(S \cup T) - (S \cap T)$ is the set of all elements that belong to S or T but not both.

§3.3. Inequalities

An algebraic inequality is a statement in which there are two algebraic expressions separated by one of the following: $\leq$, $<$, $\geq$, $>$.

Example 6: Examples of inequalities are the following:
(1) $2x + 5 < x^2$;
(2) $x^2y \geq x + 2y$.

Inequalities look very much like equations and we can work with them like equations in many ways. The next theorem uses the fact that the sum of two positive numbers is positive.

Theorem 1:
If $a \le b$ and $c \le d$ then $a + c \le b + d$.
Proof: Suppose $a \le b$ and $c \le d$.
Then $b - a \ge 0$ and $d - c \ge 0$.
$\therefore (b + d) - (a + c)$
$= (b - a) + (d - c) \ge 0$. ✋☺

Note that this doesn't work for subtraction. That is, just because

$a \le b$ and $c \le d$ it doesn't follow that

$$a - c \le b - d.$$

For example $12 \le 15$ and $3 \le 10$ but $12 - 3 \le 15 - 10$ is FALSE.

Similar results hold for the three other types of inequality. When it comes to multiplying or dividing inequalities we must be careful. We can multiply or divide both sides of an inequality by a positive number.

Theorem 2: If $a \le b$ and $x > 0$ then $ax \le bx$.
Proof: Suppose that $a \le b$ and $x > 0$.
Then $b - a \ge 0$ and so $bx - ax = (b - a)x \ge 0$.
Thus $ax \le bx$. ✋☺

The same is true for dividing by a positive number because if $x > 0$ then $1/x > 0$ and dividing by x is the same

as multiplying by $1/x$. Just remember that we must multiply or divide both sides by a positive number.

Example 7: If $3 < 5$ is $3x < 5x$?
Answer: If x is positive this is so, but suppose $x = -2$.
Is $-6 < -15$? No! It's the other way around: $-15 < -6$.

When you multiply by a negative number the inequality reverses direction. And if you don't know whether x is positive or negative you're stuck. You can't do anything without considering cases.

One of the standard things to do is to solve an inequality, that is to find the range of values of the variables for which the inequality holds. Here we consider only inequalities that involve just one variable.

Example 8: Solve the inequality $3x + 5 < 7x - 19$.
Solution: Subtract $3x + 5$ from both sides to get:

$$0 < 4x - 24, \text{ that is } 4x > 24.$$

We now divide both sides by 4 to get $x > 6$.
So the solution is $\{x \mid x > 6\}$, which means the set of all x such that $x > 6$.

Example 9: Solve the inequality $x^2 + 6 \le 5x$.
Solution: Subtract $5x$ from both sides to get:

$$x^2 - 5x + 6 \le 0.$$

Now factorise $x^2 - 5x + 6$ to get $(x - 2)(x - 3) \le 0$.
This is exactly what we might have done if we were solving the equation $x^2 + 6 = 5x$.
If $(x - 2)(x - 3) = 0$ then indeed we get $x = 2$ or 3, but what if $(x - 2)(x - 3) < 0$?
Clearly this can only happen if the factors have opposite signs. So either:

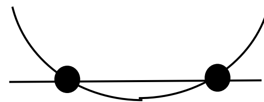
$x - 2 > 0$ and $x - 3 < 0$, in which case $2 < x < 3$,

or $x - 2 < 0$ and $x - 3 > 0$ in which case $x < 2$ and $x > 3$.

The latter case is impossible. A number can't be less than 2 and at the same time be greater than 3. So we're left with just $2 < x < 3$. Including the endpoints where we get equality and the complete solution is:

$$\{x \mid 2 \le x \le 3\}.$$

If we sketch the parabola $x^2 - 5x + 6 = (x - 2)(x - 3)$ we can see clearly what is going on.



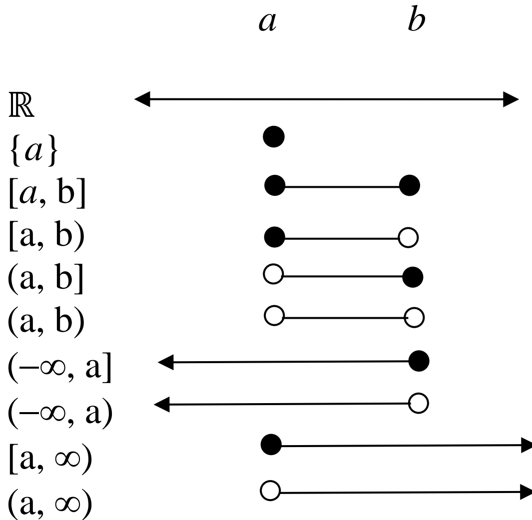
§3.4. Intervals

In each of the above cases the answer has been an interval. An **interval** on the real line is a subset S such that if x and y belong to S with $x < y$ then S contains every real number between them. The whole real line is clearly an interval, as is a set $\{a\}$ containing just one number. (The definition says that if there are two different numbers in the set the

set has to contain every number between them. It doesn't say that there have to be two distinct numbers in the set.) These are the extreme types of interval. All intervals have to be one of the following ten types:

$$\mathbb{R}$$
$$\{a\}$$
$$[a, b] = \{x \mid a \leq x \leq b\};$$
$$[a, b) = \{x \mid a \leq x < b\};$$
$$(a, b] = \{x \mid a < x \leq b\};$$
$$(a, b) = \{x \mid a < x < b\};$$
$$(-\infty, a] = \{x \mid x \leq a\};$$
$$(-\infty, a) = \{x \mid x < a\};$$
$$[a, \infty) = \{x \mid x \geq a\};$$
$$(a, \infty) = \{x \mid x > a\};$$

We can represent these by means of diagrams:



The answer to Example 8 can be written as $(6, \infty)$ and the answer to Example 9 as $[2, 3]$.

§3.5. Absolute Values

The **absolute value** of a real number x is the magnitude of x, ignoring the sign. It is denoted by $|x|$. If x is positive then $|x| = x$ and if x is negative then $|x| = -x$. And, of course $|0| = 0$.

We can define the absolute value of x very compactly as $\sqrt{x^2}$, remembering that √ denotes the positive square root. But, of course, it would be silly to find the absolute value this way!

Clearly $|xy| = |x|.|y|$ for all real numbers x, y but things don't work out quite so neatly for sums. It's NOT true in general that $|x + y| = |x| + |y|$. If x, y have opposite signs then $|x + y|$ will be less than $|x| + |y|$.

For example. $|3 + (-1)| = 2$ while $|3| + |-1| = 4$. All we can say for sums is that $|x + y| \le |x| + |y|$.

Theorem 3 (Triangle Inequality):

$|x + y| \le |x| + |y|$ for all real numbers x, y.

Proof: We could prove this by examining various cases but the simplest proof is the following.

First note that $xy = \pm|x|.|y| \le |x|.|y|$.

Hence $(x + y)^2 = x^2 + y^2 + 2xy$

$$= |x|^2 + |y|^2 + 2xy$$
$$\le |x|^2 + |y|^2 + 2|x|.|y|$$
$$\le (|x| + |y|)^2.$$

Taking positive square roots we get $|x + y| \le |x| + |y|$. ✋☺

We get equality unless x, y have opposite signs. The reason for calling this result the **Triangle Inequality** is that the corresponding result for complex numbers (where numbers live in a plane and |z| denotes the length of the line joining 0 to z) can be interpreted as saying that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.